

Algebra courses and qualifying exam

Revised June 2023

1. ALGEBRA COURSES (BOTH SEMESTERS)

Textbooks.

- *Algebra: Chapter 0*, by Paolo Aluffi. AMS Graduate Studies in Mathematics 104.
- *Algebra*, by Serge Lang. Springer GTM 211.

Lang is a classic, comprehensive and containing many insights. Aluffi's treatment is somewhat more modern, less terse but also less encyclopedic; for example there is no discussion of representation theory in Aluffi. In the syllabus below, we give references (whenever possible) to both Aluffi and Lang. There is always an advantage to learning multiple perspectives, and so we encourage students to look at both.

Prerequisites. The course will assume familiarity with undergraduate linear algebra at an abstract level, such as the book *Linear Algebra Done Right* by Sheldon Axler.

In addition, the course will assume that students have taken an undergraduate algebra class, and are familiar with the definitions of groups, rings, fields, and homomorphisms; some elementary group theory (e.g. cosets and Lagrange's theorem); and basic examples of groups (cyclic, dihedral, symmetric, and alternating groups). A student who wants to refresh the group-theoretic prerequisites in the summer before the course might look at Lang I.1–4 and/or Aluffi II.1–4,6–8.

Syllabus.

- (0) **Category theory:** Students will become familiar with the notion of a universal property, and will understand various constructions that arise throughout the course as instances of the same universal properties (such as products and coproducts) in different contexts.

References: Lang I.11 or Aluffi I.5, and examples throughout the course.

- (1) **Group theory:** Quotients and isomorphism theorems, group actions and applications, Sylow theorems, composition series, free groups.

References: Lang, I.1–8,12 or
Aluffi, II.1–9 and IV

- (2) **Rings and modules:** Polynomial rings, factorization and UFDs, modules over PIDs, chain conditions, localization.

References: Lang II, III.1–7, IV.1–4, X.1 or
Aluffi, III.1–6, V, VI.5

- (3) **Fields and Galois theory:** Field extensions, algebraic closure, Galois theory and examples, (in)solvability of polynomials, transcendence bases.

References: Lang IV.6, V, VI.1–8,13 VIII.1 or
Aluffi VII (except VII.2.2–2.3) + Lang VI.13 for the normal basis theorem

- (4) **Algebraic sets:** Hilbert's Nullstellensatz, introduction to algebraic varieties.

References: Lang IX.1–2 or
Aluffi VII.2.2–2.3

- (5) **Advanced linear algebra:** Linear algebra over rings, matrix canonical forms, bilinear maps, spectral theory of Hermitian and symmetric matrices, tensor products of modules, symmetric and alternating products.

References: Lang XIII.1–6, XIV, XV.1–8, XVI.1–2,4–8, XIX.1 or Aluffi, VI.1–3,6–7, VIII.1–4 excluding VIII.2.3–2.4

- (6) **Semisimple algebras and representation theory of finite groups:** Semisimple rings, the Artin–Wedderburn theorem, group algebras, character theory of finite groups over \mathbf{C} and examples, induced representations.

References: Lang XVII.1–6, XVIII.1–7 or Rotman, *Advanced Modern Algebra, Part 2* (GSM 180), Chapter C-2

If time permits, additional topics will be chosen at the discretion of the instructor. These may include, for instance, additional topics in commutative algebra (e.g. integrality, primary decomposition), additional topics in category theory, or an introduction to homological algebra (e.g. flatness, and the Tor and Ext functors).

We aim to cover parts (0)–(3) in the fall, and (4)–(6) plus any additional topics in the spring.

2. QUALIFYING EXAM

The emphasis of the qualifying exam will be on the topics listed in items (0)–(6) under “syllabus”, but all material covered in the prerequisites and the listed references for items (0)–(6) is potentially examinable, with one exception: while the representation theory of finite groups will be examined on the quals, the general theory of semisimple algebras will not.

In items (0)–(5), where references to both Aluffi and Lang are given, our intention is that studying *one or the other* will be sufficient to answer the questions on the qualifying exam. However, we reiterate that there is always an advantage to learning multiple perspectives, and so *we encourage students to look at both*. The same principle applies to the two references given for item (6). Important: this includes the exercises. For example, some topics covered in the text of Lang are covered in the exercises of Aluffi.

Any additional topics, covered as time permits, are not on the exam syllabus, even if covered during the course.

2.1. Additional references. An appealing reference for the representation theory of finite groups that does not pass first through the general theory of semisimple algebras is Lectures 1–3 of *Representation Theory* by Fulton and Harris (GTM 129).

Other general references that students may wish to consult include Dummit & Foote, which is thorough and includes many interesting exercises; and *Algebra* by Thomas Hungerford (GTM 73), another classic text.