Analysis courses and qualifying exam

Revised June 2023

1. First semester: real variables

Textbook. *Measure and Integral: An Introduction to Real Analysis (2nd Edition)*, by R. L. Wheeden and A. Zygmund.

Syllabus.

(1) **Knowledge of material from undergraduate analysis**: topics such as open and closed sets, compactness (incl. Heine-Borel theorem), continuity (incl. uniform continuity), uniform convergence and the Arzela-Ascoli theorem.

(2) **Measure theory**: Measurable sets and functions, outer measure, construction of Lebesgue measure. Lusin's theorem. Notions of convergence involving measure (pointwise a.e., convergence in measure). Egorov's theorem.

(3) **Lebesgue integral**: Definition of $\int_X f(x)d\mu(x)$ and relation to the distribution function $\mu\{x : f(x) > t\}$. Fatou's Lemma and dominated convergence. $L^1(X, \mu)$ as a normed space. Relation of convergence in L^1 to pointwise a.e. convergence and convergence in measure. Product measure and the Fubini theorem. Relation between Lebesgue and Riemann integrals. Riesz representation theorem concerning positive linear functionals on $C(X)$.

(4) L^p **spaces**: Jensen, Holder and Minkowski inequalities. Completeness. Duality of L^p and L^q for $\frac{1}{p} + \frac{1}{q} = 1$. Bounded functionals, weak convergence on L^p. Uniform boundedness principle for $L^p(X, \mu)$. Approximation of L^p functions on $\mathbb{R}^n$ by smooth functions. Weak compactness of unit ball in L^p. The space $L^2(X, \mu)$. Hilbert space: Bessel inequality, orthonormal bases.

Coverage. This course will be a one-semester treatment of standard real-variable theory. The goal will be to at least cover the **first nine chapters** of Wheeden and Zygmund.

2. Second semester: complex variables

Textbook. Introduction to Complex Analysis by Michael E. Taylor. (480 pages, GSM#202)

Syllabus.

(1) **Analytic (= holomorphic) functions**: Power series, radius of convergence, Cauchy-Riemann equations. Liouville theorem, Cauchy estimates. Uniform limits. Discreteness of zeros.

(2) **Meromorphic functions**: Riemann's removable singularities theorem, poles, Laurent series, Cauchy integral formula, residue theorem, residue calculus.

(3) **Local behaviour of holomorphic functions**: Argument principle, zeros of holomorphic functions, Rouché's theorem, Hurwitz's theorem. Maximum modulus principle. Open mapping theorem.

(4) **Holomorphic mappings**: Riemann sphere, linear fractional transformations, conformal mapping, Schwarz lemma, normal families, Riemann mapping theorem.

(5) **Elements of Fourier analysis**: Fourier transform on $L^1(\mathbb{R}^n)$ and $L^1(\mathbb{S}^1)$ ($\mathbb{S}^1 =$ unit circle). Riemann-Lebesgue lemma. Plancherel theorem (Fourier transform as a unitary operator on $L^2(\mathbb{R}^n)$). Parseval inequality for the Fourier tranform on $\mathbb{S}^1$. Inversion formula. Convolution. Hausdorff-Young inequality.

Coverage

This course will be a one-semester treatment of standard complex-variable theory. The goal will be to at least cover the **first five chapters** of Taylor.

3. Qualifying exam

The emphasis of the qualifying exam will be on the topics listed above under "syllabus", but all material from the first nine chapters of Wheeden and Zygmund and the first five chapters of Taylor is potentially examinable.

The real analysis and complex analysis halves of the qualifying exam are passed or failed independently. A student passing one of the halves, but not the other, only needs to retake the half they did not pass. A student taking half of the exam will have half the time.