

ANALYSIS QUAL

FALL 2025

REAL ANALYSIS

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Prove that the functions $e^{2\pi i n x}$, $n \in \mathbf{Z}$, are a complete, orthonormal basis for $L^2(\mathbf{S}^1)$, where \mathbf{S}^1 has the usual definition, $\mathbf{S}^1 = [0, 1]$, with 0 and 1 glued together.

Problem 2. Let f, g be functions mapping \mathbf{R}^n to \mathbf{R} and let $f * g$ denote the usual convolution,

$$(1) \quad (f * g)(x) = \int f(y)g(x - y)dy.$$

For any natural number m , let

$$(2) \quad g_m(x) = \left(\frac{m}{\pi}\right)^{n/2} e^{-m|x|^2}.$$

Show that for any $f \in L^2(\mathbf{R}^n)$, $g_m * f \rightarrow f$ in L^2 norm. Briefly explain why convergence in L^∞ fails. That is, $g_m * f$ need not converge to $f \in L^\infty$.

Problem 3. Let ℓ^p be the space of real valued sequences $\mathbf{c} = \{c_k\}_{k=1}^\infty$ such that $\|\mathbf{c}\|_p = \left(\sum_{k=1}^\infty |c_k|^p\right)^{1/p} < \infty$ and suppose $1 < p < \infty$. Show that

$$(3) \quad L(\mathbf{c}) = \sum_{k=1}^\infty d_k c_k,$$

defines a bounded linear functional on ℓ^p if $\mathbf{d} = \{d_k\}_{k=1}^\infty$ is in ℓ^q , where $1/p + 1/q = 1$. Conversely show that every bounded linear functional L on ℓ^p can be written as (3) for some $\mathbf{d} = \{d_k\}_{k=1}^\infty$ in ℓ^q , where $1/p + 1/q = 1$.

Problem 4. Suppose that $\{E_k\}_k$ is a countable family of measurable subsets of \mathbf{R}^n and that their Lebesgue measures satisfy

$$\sum_k |E_k| < +\infty.$$

Let

$$E = \limsup E_k = \{x \in \mathbf{R}^n : x \in E_k \text{ for infinitely many } k\}.$$

(1) Show that E is measurable.

(2) Prove that $|E| = 0$.

COMPLEX ANALYSIS

Choose three out of four problems and show all work with each problem on a new page.

Problem 5. Let $\{f_n\}$ be a sequence of holomorphic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and suppose that $f_n \rightarrow f$ uniformly on compact subsets of \mathbb{D} , where $f: \mathbb{D} \rightarrow \mathbb{C}$ is continuous. Prove that f is holomorphic on \mathbb{D} .

Problem 6. Use contour integration to evaluate

$$\int_0^\infty \frac{x^2}{x^4 + 1} dx.$$

Problem 7. Let f be an analytic function on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and suppose that

$$|f(z)| \geq \sqrt{|z|} \quad \text{for all } z \in \mathbb{D}.$$

Show that

$$|f(z)| \geq 1 \quad \text{for all } z \in \mathbb{D}.$$

Problem 8. Determine how many solutions $z^7 - 7z^4 + z^3 - 3z = 1$ are in the annulus $1 < |z| < 2$.

Analysis Qualifying Exam, Spring 2025

Name:

JHED ID:

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 - Simplify all answers as far as possible.
 - Solutions without proper justification will receive no credit.
-

Real Problems	Your score
1	
2	
3	
4	
Total	

Complex Problems	Your score
1	
2	
3	
4	
Total	

Real Analysis Part

Choose three of four problems and show all work with each problem on a new page.

1) Suppose that $f(x, t) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ satisfies

$$|f(x, t)|, \left| \frac{d}{dt} f(x, t) \right| \leq e^{-|x|^2}, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

If

$$F(t) = \int_{\mathbb{R}^n} f(x, t) dx,$$

prove that F is differentiable and

$$F'(t) = \int_{\mathbb{R}^n} \frac{d}{dt} f(x, t) dx.$$

2) Let $K \subset \mathbb{R}$ be compact, and let $\delta(z) = \inf_{x \in K} |x - z|$ denote the distance from z to K . Prove that for almost every $x \in K$

$$\delta(x + y) = o(y), \quad \text{as } y \rightarrow 0.$$

3) Suppose that $f \in L^{4/3}(\mathbb{R}^n)$ and $K \in L^4(\mathbb{R}^n)$. Show that $f * K$ is bounded and uniformly continuous.

4) Suppose that $\{a_k\} \in \ell^1(\mathbb{Z})$ and that $f \in L^2([-\pi, \pi])$ satisfies

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

Prove that f can be modified on a set of measure zero so that the resulting function, \tilde{f} , is continuous on $[-\pi, \pi]$ and satisfies $\tilde{f}(\pi) = \tilde{f}(-\pi)$.

Complex Analysis Part

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Let $f(z)$ be holomorphic on $\mathbb{C} \setminus \mathbb{R}$. f is continuous at each point of \mathbb{R} . Show that f extends to an entire analytic function. (Hint: Use Morera's theorem.)

Problem 2. Evaluate p.v. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2} dx$ by residue theory.

Problem 3. Let $f : \Omega \rightarrow \mathbb{C}$ be a non-constant holomorphic function defined on a connected open set $\Omega \subset \mathbb{C}$, and suppose that f is open (i.e., maps open sets to open sets). Additionally, suppose that for some $z_0 \in \Omega$, we have $f'(z_0) = 0$. Show that f is not injective on Ω .

Problem 4. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic function on the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and suppose that

$$|f(z)| \leq M \quad \text{for all } z \in \mathbb{D},$$

for some constant $M > 0$. Prove the estimate that for all $z \in \mathbb{D}$,

$$|f'(z)| \leq \frac{M}{(1 - |z|)^2}.$$

Johns Hopkins University, Department of Mathematics
Analysis Qualifying Exam, Spring 2024

Name: _____

Real Analysis

Problem 1

Let f_k, f be functions in $L^p(\mathbb{R}^n)$ with $1 < p < \infty$, such that $f_k \rightarrow f$ a.e.. Suppose $\|f_k\|_p \leq M < +\infty$.

1. Show that

$$\int f_k g \rightarrow \int f g \quad \text{for all } g \in L^q(\mathbb{R}^n) \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

2. Does the result still hold when $p = 1$? Justify your answer.

Problem 2

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. Show that the graph of f is a Lebesgue measurable set of measure zero in \mathbb{R}^{n+1} .

Problem 3

1. Let H_1 and H_2 be two Hilbert spaces over the real numbers in \mathbb{R} . Suppose $T : H_1 \rightarrow H_2$ is a linear isometry from H_1 to H_2 , i.e. T is linear and satisfies $\|Tx\| = \|x\|$ for all $x \in H_1$. Prove that

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \text{for all } x, y \in H_1.$$

2. Suppose that $\{\varphi_k\}_k$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Show that the collection $\{\varphi_{k,j}\}_{k,j}$ defined as $\varphi_{k,j}(x, y) = \varphi_k(x)\varphi_j(y)$ is an orthonormal basis for $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

Complex Analysis

Problem 1

Show that an holomorphic function cannot have a constant modulus without reducing to a constant.

Problem 2

How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? *Hint: Look for the biggest term when $|z| = 1$ and apply Rouché's theorem*

Problem 3

Compute using residue calculus the following integral for $|a| > 1$,

$$\int_0^{\pi/2} \frac{dx}{a + \sin^2(x)}$$

Problem 4

Consider the infinite product for α not an integer

$$\prod_{n=-\infty}^{\infty} \left(1 + \frac{z}{n + \alpha}\right) e^{-z/(n+\alpha)}.$$

1. Show that the infinite product above is well-defined.
2. Show that there exists an entire function h such that

$$\sin(\pi(n + \alpha)) = e^{h(z)} \prod_{n=-\infty}^{\infty} \left(1 + \frac{z}{n + \alpha}\right) e^{-z/(n+\alpha)}.$$

Name: _____

1 First part: real analysis

Problem 1

Let $I = [0, 1]$ and for $n \in \mathbb{N}$, consider $0 \leq j \leq 2^n - 1$. Define

$$I_{nj} = [j2^{-n}, (j+1)2^{-n}].$$

Let $f \in L^1(I)$ and define

$$E_n(f)(x) = \sum_{j=0}^{2^n-1} \left(2^n \int_{I_{nj}} f dt \right) \chi_{I_{nj}}(x).$$

Prove that $\lim_{n \rightarrow \infty} E_n(f)(x) = f(x)$ a.e. in I .

Problem 2

1. Prove the existence of a nonnegative function φ from \mathbb{R} into \mathbb{R} such that $\varphi \in C^\infty(\mathbb{R})$ with compact support and

$$\int_{\mathbb{R}} \varphi = 1.$$

2. Let φ be the function constructed previously. Define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi(\frac{x}{\varepsilon})$, $\varepsilon > 0$. Prove that if $f \in L^2(\mathbb{R})$ then the convolution $f * \varphi_\varepsilon$ is well-defined and belongs to $L^2(\mathbb{R})$. Furthermore, show that $f * \varphi_\varepsilon$ is in L^2 uniformly in ε .
3. Prove Young's inequality on the convolution of Lebesgue spaces: $L^p(\mathbb{R}^d) * L^q(\mathbb{R}^d) \subset L^r(\mathbb{R}^d)$ if $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Problem 3

Let $L^2 = L^2(\mathbb{R}^d)$ be the real Hilbert space endowed with its natural norm $\|\cdot\|$ derived from the real inner product $(f, g) = \int fg dm$ (where dm is Lebesgue measure on \mathbb{R}^d). We say that $\{f_n\} \subset L^2$ converges weakly to f if $(f_n, g) \rightarrow (f, g)$ for every $g \in L^2$.

- Prove that if f_n converges weakly to f and $\|f_n\| \rightarrow \|f\|$ then f_n converges to f in the strong topology, i.e. the topology induced by the norm.
- Prove that there exists a sequence of bounded functions in L^2 which is not converging in L^2 but weakly converging up to a subsequence possibly. What do you conclude on the unit ball of L^2 endowed with the strong topology?

Problem 4

Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function with $\phi(0) = 0$. Show that

$$\int_0^1 \phi \circ f \, dx = \int_{\mathbb{R}} |\{x \in [0, 1] : f(x) > t\}| \phi'(t) \, dt$$

2 Second part: complex analysis

Problem 1

- a) Show that if an entire function f satisfies $|f(z)| < \sqrt{|z|}$ for all $z \in \mathbb{C}$, then $f(z)$ is a constant function.
b) Why isn't the function $z^{1/2}$ a counterexample for Part a)?

Problem 2

For $a \in (-1, 0)$, evaluate the following integral using residue theory,

$$\int_0^\infty \frac{x^a}{1+x} \, dx;$$

you will justify why this improper integral exists before doing the calculation.

Problem 3

Let f be a *nonnegative harmonic* function in a neighborhood of the disk $D(P, R)$ centered at P and of radius R . Show the following Harnack inequality: for all $z \in D(P, R)$

$$\frac{R - |z - P|}{R + |z - P|} f(P) \leq f(z) \leq \frac{R + |z - P|}{R - |z - P|} f(P).$$

Using the previous Harnack inequality, prove the classical Liouville theorem: every *bounded and harmonic* function on all of \mathbb{R}^2 must be constant.

Analysis qualifying exam, complex analysis part
Spring 2023

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JHED:

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4		

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Problem 1.

Evaluate $\int_0^\infty \frac{\sin^2 x}{x^2} dx$.

Problem 2.

a). Let $R(z) = \frac{p(z)}{q(z)}$ where p and q are polynomials (thus holomorphic). Let f be holomorphic on $\mathbb{C} \setminus \{p_1, \dots, p_k\}$ and suppose that f has a pole at each of the points $\{p_1, \dots, p_k\}$. If $|f(z)| \leq |R(z)|$ for all z at which $f(z)$ and $R(z)$ are defined, prove that f is a rational function.

b). Find all the isolated singularities on $\mathbb{C} \cup \{\infty\}$ and calculate their residues for the functions $f(z) = \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}$, $\alpha\beta = 1$, $\alpha \neq \beta$.

Problem 3.

a). Let $f : U = \{z : 0 < |z| < 1\} \rightarrow \mathbb{C}$ be holomorphic and nonconstant. If there exists a sequence $\{z_n\} \subset U$ which converges to 0 and satisfies that $f(z_n) = 0$, then prove that 0 is an essential singularity of f .

b). Find all entire functions f such that $\lim_{|z| \rightarrow \infty} f(z) = \infty$.

Problem 4. Assume that f is holomorphic on $0 < r < |z-a| < R < +\infty$ and continuous on $r \leq |z-a| \leq R$. If $f(z) = 0$ at $|z-a| = R$, prove that $f(z) \equiv 0$.

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Problem 4. Assume that f is holomorphic on $0 < r < |z-a| < R < +\infty$ and continuous on $r \leq |z-a| \leq R$. If $f(z) = 0$ at $|z-a| = R$, prove that $f(z) \equiv 0$.

Qualifying Exam in Analysis, Fall 2022, August 30, 12-3pm

Part I: Real Analysis

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Prove or disprove that if $\{f_n\}_{n=1}^\infty$ is a sequence of Lebesgue integrable functions $f_n : [0, 1] \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} \|f_n\|_{L^1} \rightarrow 0$, then for at least one value $x \in [0, 1]$, we have $\lim_{n \rightarrow \infty} f_n(x) = 0$.

Problem 2. Let $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$. Show that

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2} dt < \infty.$$

Problem 3. Let $f \in L^1(\mathbb{R}^n)$, and let $|\{x \in \mathbb{R}^n : |f(x)| \geq \lambda\}|$ denote the Lebesgue measure of the set where $|f| \geq \lambda$. Prove that

$$\|f\|_{L^1} = \int_0^\infty |\{x : |f(x)| \geq \lambda\}| d\lambda.$$

Problem 4. Let $f \in L^1(\mathbb{R}^n)$. Evaluate

$$\lim_{y \rightarrow \infty} \int_{\mathbb{R}^n} |f(x+y) - f(x)| dx.$$

Note that the limit is NOT as $y \rightarrow 0$.

Part II: Complex Analysis

Choose three of four problems and show all work with each problem on a new page.

Problem 1. For $-1 < a < 0$, evaluate

$$\int_0^{\infty} \frac{x^a}{1+x} dx$$

by residue theory.

Problem 2. Let f be holomorphic and one-to-one on an open set $\Omega \subset \mathbb{C}$. Show that $f'(z) \neq 0$ for all $z \in \Omega$.

Problem 3. Let $f : D(0, 1) \rightarrow \mathbb{C}$ be a non-constant holomorphic function which satisfies $\operatorname{Re}f(z) \geq 0$ for all $z \in D(0, 1)$. Here $D(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk.

- (1) Show that $\operatorname{Re}f(z) > 0$ for all $z \in D(0, 1)$.
- (2) If $f(0) = a \in \mathbb{R}_+$, show that

$$a \frac{1-|z|}{1+|z|} \leq |f(z)| \leq a \frac{1+|z|}{1-|z|}.$$

Problem 4. Let $D(0, 1) \subset \mathbb{C}$ be the unit disk and f be continuous on $\partial D(0, 1)$. Show that

$$u(z) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1-|z|^2}{|z-e^{i\theta}|^2} d\theta & \text{for } z \in D(0, 1) \\ f(z) & \text{for } z \in \partial D(0, 1) \end{cases}$$

is harmonic on $D(0, 1)$ and continuous up to the boundary $\partial D(0, 1)$.

Qualifying Exam in Analysis, May 10, 2022, 12-3pm

Part I: Real Analysis

Choose three of four problems and show all work with each problem on a new page.

Q1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Show that the graph of f is a Lebesgue measurable subset of \mathbb{R}^2 of measure zero.

Q2. Suppose that the sequence $f_n \in L^3(\mathbb{R}^3)$ converges almost everywhere to f . Suppose further that

$$\|f_n\|_{L^3(\mathbb{R}^3)} \leq 3.$$

Prove that $f \in L^3(\mathbb{R}^3)$ and that

$$\|f\|_{L^3(\mathbb{R}^3)} \leq 3.$$

Q3. Let $f \in L^1(\mathbb{R})$ and let g be a bounded measurable function on \mathbb{R} . Recall that the convolution of f and g is the function $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) dy.$$

Prove that the convolution $f * g$ is well-defined, bounded and uniformly continuous on \mathbb{R} .

Q4. Let $f \in L^1(\mathbb{R}^d)$. Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\left| \int_E f(y) dy \right| < \varepsilon$$

whenever $m(E) < \delta$.

Part II: Complex Analysis

Choose three of four problems and show all work with each problem on a new page.

Q1. Using residue theory, evaluate

$$\int_0^{\infty} \frac{x^{\frac{1}{2}}}{x^2 + x + 1} dx.$$

Q2. Let f be holomorphic on $D(0, 1)$ and continuous on $\overline{D}(0, 1)$. Suppose $|f(z)| = 1$ for $|z| = 1$. Prove that f must be a rational function.

Q3. Let $f : D(0, 1) \rightarrow D(0, 1)$ be a holomorphic function satisfying

$$f(0) = f'(0) = \dots = f^{(n-1)}(0) = 0.$$

Show that

- 1) $|f(z)| \leq |z|^n$ for $z \in D(0, 1)$;
- 2) If either there is some $P \in D(0, 1) \setminus \{0\}$ such that $|f(P)| = |P|^n$ or $|f^{(n)}(0)| = n!$, then $f(z) = e^{i\theta} z^n$ for some $\theta \in \mathbb{R}$.

Q4.

- 1) Let f be a non-negative harmonic function on a neighborhood of $\overline{D}(P, R)$. Show that

$$\frac{R - |z - P|}{R + |z - P|} f(P) \leq f(z) \leq \frac{R + |z - P|}{R - |z - P|} f(P) \quad \forall z \in D(P, R).$$

- 2) By using the Harnack inequality in 1), prove the following Liouville theorem:
Let f be harmonic on all of \mathbb{C} . If f is bounded from below, then f must be a constant.

Qualifying Exam in Analysis, September 8, 2021, 12-3pm

Part I: Real Analysis

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions $f_n : [0, 2\pi] \rightarrow \mathbb{C}$, $n = 1, 2, \dots$. Suppose there exists $f \in L^2([0, 2\pi])$ such that $f_n \rightarrow f$ in L^2 , i.e.,

$$\int_{[0, 2\pi]} |f_n(x) - f(x)|^2 dx \rightarrow 0.$$

a) Show that $f_n \rightarrow f$ in measure, that is,

$$\lim_{n \rightarrow \infty} |\{x \in [0, 2\pi] : |f_n(x) - f(x)| \geq \varepsilon\}| = 0$$

for all $\varepsilon > 0$, with $|A|$ denoting the Lebesgue measure of a set A .

b) Show that there is a subsequence f_{n_k} such that $f_{n_k}(x) \rightarrow f(x)$ almost everywhere in $[0, 2\pi]$. (Do not just quote a theorem.)

Problem 2. For $f \in L^2(\mathbb{R})$ let

$$\mathcal{I}(f) = \int_{\mathbb{R}} \frac{f(x)}{x^{1/3}(1+|x|)^{2/3}} dx.$$

Is $\mathcal{I}(f)$ always finite? Give a proof or counterexample.

Problem 3. Suppose $\varphi \in L^1(\mathbb{R})$ with $\int \varphi(x) dx = a$ and for $\delta > 0$ define

$$\mathcal{A}_\delta f(x) = \int f(x-y) \delta^{-1} \varphi(y/\delta) dy.$$

a) Prove that

$$\|\mathcal{A}_\delta f\|_{L^1} \leq \|\varphi\|_{L^1} \|f\|_{L^1}$$

for each $\delta > 0$.

b) Prove that $\mathcal{A}_\delta f \rightarrow a \cdot f$ in $L^1(\mathbb{R})$ as $\delta \searrow 0$.

Problem 4. Let $f \in L^2(\mathbb{R})$. If

$$\int_{[a, b]} f(x) dx = 0$$

for all rational numbers a and b with $a < b$, prove that $f(x) = 0$ almost everywhere.

Part II: Complex Analysis

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Show that there does not exist a solution to

$$\Delta u = u^3$$

on \mathbb{R}^d that is positive everywhere and satisfies $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Problem 2. Prove that for $0 < \operatorname{Re}(z) < 1$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

when

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Problem 3. Let $f(x) = e^{-\frac{x^2}{2}}$. Compute the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} dx.$$

Problem 4. Describe all the fractional linear transformations that map $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R} \cup \{\infty\}$.

Qualifying Exam in Analysis, May 12, 2021, 12-3pm

Part I: Real Analysis

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} f(x + \sqrt{k})$$

converges absolutely for almost every $x \in \mathbb{R}$.

Problem 2. For $f \in C_0^\infty(\mathbb{R}^3)$ define

$$u(x, t) = \int_{\mathbb{R}^3} e^{2\pi i x \cdot \xi} \frac{\sin t|\xi|}{|\xi|} f(\xi) d\xi, \quad x \in \mathbb{R}^3, t > 0.$$

Show that $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^2(\mathbb{R}^3)} = \infty$ for a set of f that is dense in $L^2(\mathbb{R}^n)$.

Problem 3. Consider the function on \mathbb{R} defined by

$$\varphi(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 0, & x \notin [0, 1], \end{cases}$$

and for $\delta > 0$ set $\varphi_\delta(x) = \delta^{-1}\varphi(x/\delta)$. Prove that if $f \in L^1(\mathbb{R})$ then

$$(\varphi_\delta * f)(x) \rightarrow \frac{1}{2}f(x) \quad \text{for almost every } x \in \mathbb{R},$$

if $\varphi_\delta * f$ denotes convolution.

Problem 4. Suppose that $\{\phi_k\}$ is an orthonormal system of continuous functions in $L^2([0, 1])$ and let S denote the closure of the span of $\{\phi_k\}$. If

$$\sup_{f \in S \setminus \{0\}} \frac{\|f\|_{L^\infty([0,1])}}{\|f\|_{L^2([0,1])}}$$

is finite, prove that S is finite dimensional.

Part II: Complex Analysis

Choose three of four problems and show all work with each problem on a new page.

Problem 1. Suppose $U \subset \mathbb{C}$ is an open set. Suppose f and g are functions holomorphic on $U \subset \mathbb{C}$. Show that if

$$|f(z)|^4 + |g(z)|^4$$

attains a maximum for some $z_0 \in U$, then f and g are constant on U .

Problem 2. Suppose f is holomorphic on $\{z : 0 < |z| < 1\}$, and that

$$|f(z)| \leq \left(\log \frac{1}{|z|}\right)^{80} \quad \text{in} \quad \left\{z : |z| \leq \frac{1}{2}\right\}.$$

Show that f has a removable singularity at $z = 0$.

Problem 3. Let $f(x) = \frac{1}{x^2+1}$. Use a contour integral consisting of the interval $[-R, R] \subset \mathbb{R}$ and a semicircle of radius R to compute the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} dx.$$

Problem 4. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D \rightarrow D$ be a holomorphic function. Prove

$$\frac{|f(z)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

Name: _____ Date: _____

Problem 1

Let $f \in L^p(\mathbb{R}^d)$. Prove that

$$\lim_{h \rightarrow 0} \|f(\cdot - h) - f(\cdot)\|_{L^p} = 0.$$

Problem 2

1. Prove the existence of a nonnegative function φ from \mathbb{R} into \mathbb{R} such that $\varphi \in C^\infty(\mathbb{R})$ with compact support and

$$\int_{\mathbb{R}} \varphi = 1$$

2. Let φ be the function constructed previously. Define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi(\frac{x}{\varepsilon})$, $\varepsilon > 0$. Prove that if $f \in L^2(\mathbb{R})$ then the convolution $f * \varphi_\varepsilon$ is well-defined and belongs to $L^2(\mathbb{R})$. Furthermore, show that $f * \varphi_\varepsilon$ is in L^2 uniformly in ε .
3. Prove Young's inequality: $L^p * L^q \subset L^r$ if $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Problem 3

Prove that if $f \in L^1(\mathbb{R}^d)$ then its Fourier transform $\hat{f}(\xi)$ is such that $\hat{f}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$. Use this result to show that there is no unit element in $L^1(\mathbb{R}^d)$ for the operation defined by convolution; i.e there is no $h \in L^1(\mathbb{R}^d)$ such that for all $f \in L^1(\mathbb{R}^d)$, we have $f * h = f$.

Problem 4

Compute the integrals

- 1) $\frac{1}{2\pi i} \oint_{\gamma} \frac{e^z}{z(z+2)^2} dz$, where γ is the negatively oriented triangle with vertices $1 \pm i$ and -3 .
- 2) p.v. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$.

Problem 5

- 1). Evaluate $\int_0^1 \frac{1}{x^2+4} \sqrt{\frac{x}{1-x}} dx$.
- 2). Let $f(z) = e^{z+\frac{1}{z}}$. Prove that $Res_f(0) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}$.

Problem 6

Let $f(z)$ be an entire function satisfying $|f(z)| \leq |f(z^2)|$ for all z . Show that $\frac{1}{f(z)}$ is bounded and is an entire function, unless $f(z) \equiv 0$.

Name: _____ Date: _____

Problem 1

Let $I = [0, 1]$ and for $n \in \mathbb{N}$, consider $0 \leq j \leq 2^n - 1$. Define

$$I_{nj} = [j2^{-n}, (j+1)2^{-n}].$$

Let $f \in L^1(I)$ and define

$$E_n(f)(x) = \sum_{j=0}^{2^n-1} \left(2^n \int_{I_{nj}} f dt \right) \chi_{I_{nj}}(x).$$

Prove that $\lim_{n \rightarrow \infty} E_n(f)(x) = f(x)$ a.e. in I .

Problem 2

Let $L^2 = L^2(\mathbb{R}^d)$ be the real Hilbert space endowed with its natural norm $\|\cdot\|$ derived from the real inner product $(f, g) = \int fg dm$ (where dm is Lebesgue measure on \mathbb{R}^d). We say that $\{f_n\} \subset L^2$ converges weakly to f if $(f_n, g) \rightarrow (f, g)$ for every $g \in L^2$.

- Prove that if f_n converges weakly to f and $\|f_n\| \rightarrow \|f\|$ then f_n converges to f in the strong topology.
- Prove that there exists a sequence of bounded functions in L^2 which is not converging in L^2 but weakly converging up to a subsequence possibly. What do you conclude on the unit ball of L^2 endowed with the strong topology?

Problem 3

Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function with $\phi(0) = 0$. Show that

$$\int_0^1 \phi \circ f dx = \int_{\mathbb{R}} |\{x \in [0, 1] : f(x) > t\}| \phi'(t) dt$$

Problem 4

Compute the integrals

$$a) \int_{-\infty}^{\infty} \frac{x^4}{x^{10} + 1} dx; \quad b) \text{ p.v. } \int_{-\infty}^{\infty} \frac{x+2}{x^4 + 3x^3 + x^2 + 3x} dx.$$

Problem 5

Show that if $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function such that $f(\mathbb{C}) \cap \{ix : x \in \mathbb{R}, x \geq 0\} = \emptyset$, then f is a constant.

Problem 6

Let U be a simply connected domain in \mathbb{C} . Let u_n be a sequence of holomorphic functions on U such that

$$\int_U |u_n(z) - u(z)| dz \rightarrow 0.$$

Prove that u is a holomorphic function on U .

Qualifying Exam in Analysis, September 4, 2019, 9am-12noon.

Part I: Real Analysis

Choose three of four problems and show all work with each problem on a new page.

1. Assume that f_1, f_2, \dots is a sequence of positive continuous functions defined on $[0, 1]$ with

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ for every } x \in [0, 1]$$

and

$$\int_0^1 f_n(x) dx = 1.$$

(a) Is it always true that $\int_0^1 f(x) dx \leq 1$? Provide a proof if it is true or provide a counter example if it is false.

(b) Is it always true that $\int_0^1 f(x) dx \geq 1$? Provide a proof if it is true or provide a counter example if it is false.

2. Prove that if a sequence f_1, f_2, \dots of functions in $L^1(\mathbb{R})$ converges to f in $L^1(\mathbb{R})$, then there exists a subsequence f_{n_1}, f_{n_2}, \dots that converges a.e. to f .

3. Define

$$f * g(x) := \int_{-\infty}^{\infty} f(y)g(x - y) dy.$$

Prove that if $f, g \in L^2(\mathbb{R})$, then $f * g$ is a continuous function on \mathbb{R} .

4. Prove that every closed convex subset of a Hilbert space has a unique element of minimal norm.

Part II. Complex Analysis

Choose three of four problems and show all work with each problem on a new page.

1. Evaluate $\int_0^\infty \frac{\log x}{x^2+2} dx$ by contour integration using the positively oriented contour from $-R$ to R on the real axis (indented at the origin) and the positively oriented semicircle $|z| = R$, $\text{Im}z > 0$. Choose an appropriate branch of logarithm.

2. Let $f(z)$ be an entire function such that

$$\max_{|z|=R} |f(z)| \leq AR^k + B,$$

for positive constants A, B and all $R > 1000$. Show that f is a polynomial of degree at most k .

3a. Define what is meant by a normal family of holomorphic functions on an open (possibly unbounded) domain U .

b. Suppose \mathcal{F} is a normal family of holomorphic functions on the open unit disk D . Show that the family

$$\mathcal{F}' = \{f' : f \in \mathcal{F}\}$$

is also a normal family on D .

4. Let f be a holomorphic function in the punctured disk $\{z : 0 < |z| < 2\}$ satisfying

$$|f(z)| \leq \left(\log \frac{1}{|z|}\right)^{100} \text{ in } \{|z| \leq 1/2\},$$

$$|f(z)| = 1 \text{ on } |z| = 1.$$

a. Show that f has a removable singularity at the origin.

b. Show that if $f(z) \neq 0$ in $|z| < 1$, then f is constant.

c. (Extra credit) True or false, explain.

$f = \alpha z^n$ for $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and an integer $n \geq 0$.

ANALYSIS QUALIFYING EXAM: SPRING 2019

Please answer three questions from the real analysis section and three questions from the complex analysis section.

1. REAL ANALYSIS

Question 1.1. Suppose that $f_j \in L^2(\mathbb{R}^d)$, $j = 1, 2, \dots$, and $f \in L^2(\mathbb{R}^d)$ satisfy

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^d} f_j g = \int_{\mathbb{R}^d} f g$$

for all $g \in L^2(\mathbb{R}^d)$. That is, f_j converges to f weakly in L^2 . Suppose that the sequence satisfies the uniform bound

(A)
$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^d |f_j(x)| \leq M < \infty.$$

Show that $\|f_j\|_2 \rightarrow \|f\|_2$ and conclude that $\|f_j - f\|_2 \rightarrow 0$. That is f_j converges to f strongly in $L^2(\mathbb{R}^d)$. Show by example that condition (A) is necessary.

Question 1.2. Fix a measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and, for every $x, y \in \mathbb{R}$, let

$$f_x : \mathbb{R} \rightarrow \mathbb{R} \text{ and } f_y : \mathbb{R} \rightarrow \mathbb{R}$$

be given by $f_x(z) = f(x, z)$ and $f_y(z) = f(z, y)$. Show that there exists such an f so that $f_x \in L^1(\mathbb{R})$ for a.e. x and $f_y \in L^1(\mathbb{R})$ for a.e. y but

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_x(y) dy \right) dx \neq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_y(x) dx \right) dy.$$

What does Fubini's theorem imply about such f ? What about Tonelli's theorem?

Question 1.3. Let $f_i : [0, 1] \rightarrow \mathbb{R}$, $i = 1, 2, \dots$, be an increasing sequence of continuous functions that is uniformly bounded, i.e., for all $x \in [0, 1]$ and $i \geq 1$, $f_i(x) \leq f_{i+1}(x) \leq M < \infty$ for some fixed M . Show that $\lim_{i \rightarrow \infty} f_i(x) = g(x)$ is continuous if and only if the f_i converge uniformly to g .

Question 1.4. Show that $\lim_{t \rightarrow \infty} \int_1^t \frac{\sin(x)}{x} dx$ exists. Does $f(x) = \frac{\sin(x)}{x}$ belong to $L^1((1, \infty))$?

2. COMPLEX ANALYSIS

Question 2.1. Determine all holomorphic automorphisms of the upper half plane $\mathcal{U} = \{z : \text{Im}z > 0\}$.

Question 2.2. Let $f(z)$ be holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and continuous on \mathbb{C} . Show that f extends to an entire analytic function.

Question 2.3. Evaluate $\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x} dx$.

Question 2.4. Show that the punctured unit disk $\{z : 0 < |z| < 1\}$ and the annulus $\{z : 1 < |z| < 2\}$ cannot be conformally equivalent.

Johns Hopkins University, Department of Mathematics
Real and Complex analysis

Name: _____ Date: _____

Problem 1

Let $I = [0, 1]$ and for $n \in \mathbb{N}$, consider $0 \leq j \leq 2^n - 1$. Define

$$I_{nj} = [j2^{-n}, (j+1)2^{-n}].$$

Let $f \in L^1(I)$ and define

$$E_n(f)(x) = \sum_{j=0}^{2^n-1} \left(2^n \int_{I_{nj}} f dt \right) \chi_{I_{nj}}(x).$$

Prove that $\lim_{n \rightarrow \infty} E_n(f)(x) = f(x)$ a.e. in I .

Problem 2

Prove that the unit ball of L^2 endowed with its natural strong topology is not compact.

Problem 3

Prove that a normed vector space $(X, \|\cdot\|)$ is Banach if and only if every normally (sometimes called also absolutely) convergent series is convergent.

Problem 4

Suppose that f, g are entire functions with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there is a constant $c \in \mathbb{C}$ such that $f = cg$.

Problem 5

This problem is about the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

- Show directly that I is a convergent improper Riemann integral.
- Use a contour integral to evaluate I .

Problem 6

Let f and g be functions holomorphic defined on a domain $U \subseteq \mathbb{C}$. Set $\varphi(z) = |f(z)| + |g(z)|$ for $z \in U$. If φ assumes a maximum value on U , show that both f and g are constants on U .

Problem 7

Let $U \subseteq \mathbb{C}$ be an open set and

$$A^2(U) = \{f \text{ holomorphic on } U : \int_U |f(z)|^2 dx dy < \infty\}.$$

Define

$$(f, g) = \int_U f(z) \overline{g(z)} dx dy, \quad \forall f, g \in A^2(U).$$

Prove that $A^2(U)$ is a Hilbert space when equipped with this inner product.

Name: _____ Date: _____

Problem 1

Let $I = [0, 1]$ and for $n \in \mathbb{N}$, consider $0 \leq j \leq 2^n - 1$. Define

$$I_{nj} = [j2^{-n}, (j+1)2^{-n}].$$

Let $f \in L^1(I)$ and define

$$E_n(f)(x) = \sum_{j=0}^{2^n-1} \left(2^n \int_{I_{nj}} f dt \right) \chi_{I_{nj}}(x).$$

Prove that $\lim_{n \rightarrow \infty} E_n(f)(x) = f(x)$ a.e. in I .

Problem 2

Let $L^2 = L^2(\mathbb{R}^d)$ be the real Hilbert space endowed with its natural norm $\|\cdot\|$ derived from the real inner product $(f, g) = \int fg dm$ (where dm is Lebesgue measure on \mathbb{R}^d). We say that $f_n \in L^2$ converges weakly to f if $(f_n, g) \rightarrow (f, g)$ for every $g \in L^2$.

- Prove that if f_n converges weakly to f and $\|f_n\| \rightarrow \|f\|$ then f_n converges to f in the strong topology.
- Prove that there exists a sequence of bounded functions in L^2 which is not converging in L^2 but weakly converging up to a subsequence possibly. What do you conclude on the unit ball of L^2 endowed with the strong topology ?

Problem 3

Let $I = [0, 1]$ and denote $\|\cdot\|_p$ the p -norm $\|f\|_p = \left(\int_I |f|^p \right)^{1/p}$ for $1 \leq p < \infty$ (we admit this is a norm) and $\|f\|_\infty = \text{ess sup } |f|$.

- Show that the space of continuous functions on I endowed with the norm $\|\cdot\|_p$ for $1 \leq p < \infty$ is not a Banach space.
- Prove that the space of (Lebesgue) measurable functions on I such that their p -norm is finite is a Banach space for $1 \leq p \leq \infty$.
- Prove that there is no smooth function h such that $f * h = f$ for every $f \in L^1(I)$.
- Prove the Hölder inequality: for $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_I fg \leq \|f\|_p \|g\|_q$$

One can use the inequality $ab \leq \frac{a^p}{p} + \frac{a^q}{q}$ for any $a, b \geq 0$.

- Deduce the Young inequality: $L^p * L^q \subset L^r$ for $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Problem 4

Let f be an entire function. Suppose that for each $z_0 \in \mathbb{C}$, the power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

has at least one coefficient $c_n = 0$. Show that f is a polynomial.

Problem 5

Let U be an open subset of \mathbb{C} . Let z_0 be a point in U , and suppose that f is a meromorphic function on U with a pole at z_0 . Prove that there is no holomorphic function $g : U \setminus \{z_0\} \rightarrow \mathbb{C}$ such that $e^{g(z)} = f(z)$ for all $z \in U \setminus \{z_0\}$.

Problem 6

Suppose f is holomorphic in an annulus $r < |z| < R$, and there exists a sequence of holomorphic polynomials p_n converging to f uniformly on compact subset of the annulus. Show that f can be extended to the disc $\{|z| < R\}$ as a holomorphic function.

Problem 7

Let U be an open subset of \mathbb{C} . We use the notion

$$\|f\|_{L^2(U)} = \left(\int_U |f|^2 dx dy \right)^{1/2}.$$

- Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function. Show that for any compact set $K \subset U$, there is a constant C_K , such that

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{L^2(U)}.$$

- Prove that $\{f \text{ is holomorphic on } U : \|f\|_{L^2(U)} \leq 1\}$ is a normal family.
- Suppose U is the punctured disc $D(0, 1) - \{0\}$. If f is holomorphic on U and $\|f\|_{L^2(U)} < \infty$, prove that $z = 0$ is a removable singularity of f .

Qualifying Exam - Analysis-Fall 2017

12:30-3:30pm, Sept 8, 2017

1. Let f_n be a sequence of continuous functions on \mathbb{R} satisfying $0 \leq f_n \leq f_{n+1} \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Show that if f is continuous at x , then for any $\epsilon > 0$ there exist δ and N such that $|f_n(y) - f_n(x)| < \epsilon$ whenever $|y - x| < \delta$ and $n > N$.

2. Let $f \in L^p(\mathbb{R}^n)$. Show that

$$\lim_{h \rightarrow 0} \|f(x - h) - f(x)\|_{L^p} = 0.$$

3. For a Radon measure μ , with $\int_{\mathbb{R}^n} d\mu = C$. Prove that for all $\epsilon > 0$, there exists a set $E_\epsilon \subset \mathbb{R}^n$ s.t. $\mathcal{M}^1(E_\epsilon) := \inf_{E_\epsilon \subset \cup B_i} \{\sum_i \text{diam} B_i\} < 10\epsilon$ and for any $x \notin E_\epsilon$, $r > 0$

$$\int_{B_r(x)} d\mu \leq \frac{Cr}{\epsilon}.$$

(Hint: use Vitali covering lemma.)

4. Let $f(z)$ be a holomorphic function on $D := \{z \in \mathbb{C} : |z| < 1\}$, $|f(z)| < 1$, $f(\alpha) = 0$ for some $|\alpha| < 1$. Show that for $z \in D$

$$|f(z)| \leq \left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right|.$$

5. Let f be an entire function $|Re(f(z))| \leq C(1 + |z|)^p$ for some $p > 0$, $C > 0$. Show that f is a polynomial.

6. Let u be a subharmonic function defined on \mathbb{C} . Let $M(r) := \max_{|z|=r} u(z)$. Prove that

$$u(z) \leq \frac{\log r_2 - \log |z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log |z| - \log r_1}{\log r_2 - \log r_1} M(r_2)$$

for $0 < r_1 \leq |z| \leq r_2$.

Qualifying Exam - Analysis - Spring 2017

Justify your answers to all problems.

1. Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a measurable function and $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a monotonic, absolutely continuous function on $[0, T]$ for every $T < \infty$. Assume $\varphi(0) = 0$. Prove

$$\int_{\mathbb{R}} \varphi \circ f \, dx = \int_0^{\infty} m(\{x : f(x) > t\}) \varphi'(t) \, dt.$$

2. Let \mathcal{H} be a Hilbert space equipped with an inner product (\cdot, \cdot) and a norm $\|\cdot\| = (\cdot, \cdot)^{\frac{1}{2}}$. Recall the following: A sequence $\{f_k\} \subset \mathcal{H}$ is said *converge* to $f \in \mathcal{H}$ if $\|f_k - f\| \rightarrow 0$. A sequence $\{f_k\} \subset \mathcal{H}$ is said *converge weakly* to $f \in \mathcal{H}$ if $(f_k, g) \rightarrow (f, g)$ for any $g \in \mathcal{H}$. Prove the following statements:

(a) $\{f_k\}$ converges to f if and only if $\|f_k\| \rightarrow \|f\|$ and $\{f_k\}$ converges weakly to f .

(b) If \mathcal{H} is a finite dimensional Hilbert space, then the weak convergence implies convergence. Give a counter example to show that weak convergence does not necessarily imply convergence in an infinite dimensional Hilbert space.

(c) If a sequence $\{f_k\}$ converges weakly to f , then there exists a subsequence $\{f_{k_n}\}$ such that

$$\frac{f_{k_1} + \cdots + f_{k_n}}{n}$$

converges to f . (You may use the fact that a weakly convergent sequence is a bounded sequence.)

3. Let $\{E_k\}$ be a sequence of (Lebesgue) measurable sets in \mathbb{R}^k such that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Prove that almost every $x \in \mathbb{R}^k$ lie in at most finitely many sets E_k .

4. Let $U \subset \mathbb{C}$ be an open set, $D = \{z \in \mathbb{C} : |z| < 1\}$ and \mathcal{F} be the set of all holomorphic functions $f : U \rightarrow D$. Given $z_0 \in U$, show that there exists $f_0 \in \mathcal{F}$ such that

$$|f_0''(z_0)| = \sup_{f \in \mathcal{F}} |f''(z_0)|.$$

5. Describe all holomorphic functions on $\mathbb{C} \setminus \{0\}$ with the property that

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^{\frac{1}{2}}}, \quad \forall z \in \mathbb{C} \setminus \{0\}.$$

6. Let $f : U \rightarrow \mathbb{C}$ be a non-constant holomorphic function where $U \subset \mathbb{C}$ is an open set containing the closure \overline{D} of the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. If $|f(z)| = 1$ for all $z \in \partial D$, then prove that $D \subset f(\overline{D})$.

Qualifying Exam - Analysis - Fall 2016

Justify your answers to all problems.

1. Assume $f, f_j \in L^2([0, 1])$ for $j = 1, 2, \dots$ and $\|f_j - f\|_{L^2} \rightarrow 0$. Prove there exists a subsequence $\{f_{j'}\} \subset \{f_j\}$ such that $f_{j'} \rightarrow f$ a.e.
2. Suppose A is a Lebesgue measurable set in \mathbb{R} with $m(A) > 0$. Does there exist a sequence $\{x_n\}_{n=1}^{\infty}$ such that the complement of $\bigcup_{n=1}^{\infty} (A + x_n)$ in \mathbb{R} has measure 0? Justify your answer. (We define $A + x_n = \{a + x_n \in \mathbb{R} : a \in A\}$.)
3. Let \mathcal{H} be an infinite dimensional Hilbert space. Determine if the following statements are true or false. If true, provide a proof. If false, provide a counter example.
 - (a) A sequence $\{f_n\}$ in \mathcal{H} with $\|f_n\| = 1$ for all n has a subsequence that converges in \mathcal{H} .
 - (b) A sequence $\{f_n\}$ in \mathcal{H} with $\|f_n\| = 1$ for all n has a subsequence that converges weakly in \mathcal{H} .
4. Prove that if a sequence of harmonic functions on the open disk converges uniformly on compact subset of the disk, then the limit is harmonic.
5. Let f be a one-to-one analytic function defined on the unit disk D centered at the origin and $f(0) = 0$. Show that the function $g(z) = \sqrt{f(z^2)}$ has a single-valued branch and is also one-to-one.
6. Let $U \subset \mathbb{C}$ be an open set containing the closure \overline{D} of a unit disk. If a sequence $\{f_n : U \rightarrow \mathbb{C}\}$ of holomorphic functions converges uniformly to f on compact subsets of U , then show that there exists an integer N such that f and f_n have the same number of zeros in D for $n \geq N$.

QUALIFYING EXAM SPRING 2016 - ANALYSIS

1. Prove the absolute continuity of the Lebesgue integral; in other words, prove that if f is integrable on \mathbb{R}^d , then for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f| < \epsilon \text{ whenever } m(E) < \delta.$$

2. Prove that the Hardy-Littlewood maximal function f^* for an integrable function f satisfies

$$m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

where $\alpha > 0$. Recall that

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| dy, \quad x \in \mathbb{R}^d$$

where the supremum is taken over all balls containing the point x . You may assume the Vitali 3-times Covering Lemma. *State it clearly if you use it.*

3. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function with $\phi(0) = 0$. Prove

$$\int_0^1 \phi \circ f \, dx = \int m(\{x \in [0, 1] : f(x) > t\}) \phi'(t) dt$$

4. Let $U \subset \mathbb{C}$ be an open set and

$$A^2(U) = \{f \text{ holomorphic on } U : \int_U |f(z)|^2 dx dy < \infty\}.$$

Define

$$\langle f, g \rangle = \int_U f(z) \overline{g(z)} dx dy, \quad \forall f, g \in A^2(U).$$

Prove that $A^2(U)$ is a Hilbert space when equipped with this inner product.

5. Let $f : D \rightarrow D$ be a holomorphic function where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. Prove that if f has at least 2 fixed points then f is the identity map. (Note: A point a is said to be a fixed point of f if $f(a) = a$.)

6. Assume that $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function, not identically equal to 0 and let $\mathcal{Z} = \{z \in \mathbb{C} : f(z) = 0\}$. Prove that if \mathcal{Z} is unbounded, then f has an essential singularity at ∞ .

7. Determine the number of zeroes of the polynomial

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leq |z| \leq 2$.

QUALIFYING EXAM - ANALYSIS - FALL 2015

Justify your answers to all problems.

Notation: \mathbb{R} is the real line, \mathbb{C} is the complex plane and $D(P, r) \subset \mathbb{C}$ is the disk of radius r centered at point P .

1. Suppose $\{f_n\}_{n=1}^\infty \subset L^2(\mathbb{R})$ is a sequence that converges to 0 in the L^2 norm; in other words,

$$\|f_n\|_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f_n|^2 dx \right)^{\frac{1}{2}} \rightarrow 0.$$

Prove that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow 0$ almost everywhere.

2. Determine whether the following statements are true and false. If true, provide a proof. If false, prove a counter example.

(a) If $f(x)$ is an increasing, continuous function on the interval $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$, then there exists a set $E \subset [0, 1]$ of positive measure such that $f'(x) > 0$.

(b) If $f(x)$ is a strictly increasing, absolutely continuous function on the interval $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$, then the set $f^{-1}(E) \cap \{x \in [0, 1] : f'(x) > 0\}$ is measurable for any measurable set $E \subset [0, 1]$.

3. Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(\mathbb{R}^d)$ and define $\varphi_{k,j}(x, y) = \varphi_k(x)\varphi_j(y)$. Prove that $\{\varphi_{k,j}\}_{k,j=1}^\infty$ is an orthonormal basis of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

4. Let $U \subset \mathbb{C}$ be an open set containing $\overline{D(P, r)}$. Prove that if $f : U \rightarrow \mathbb{C}$ is a holomorphic function such that f is nowhere zero on $\partial D(P, r)$ and $g : U \rightarrow \mathbb{C}$ is a holomorphic function sufficiently uniformly close to f on $\partial D(P, r)$, then the number of zeros of f in $D(P, r)$ equals the number of zeros of g in $D(P, r)$ (counting multiplicity).

5. If $f = u + iv$ is an entire function with the property that $u(z) \leq 0$ for all $z \in \mathbb{C}$, what can you say about f ?

6. If $D(0, 1) \rightarrow \mathbb{C}$ is a function such that f^2 and f^3 are both holomorphic, prove f is holomorphic.

7. Compute the integral

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx.$$

Qualifying Exam - Analysis - Spring 2015

Justify your answers to all problems.

1. Assume $f, f_j \in L^2(\mathbb{R}^n)$ for $j = 1, 2, \dots$, $f_j \rightarrow f$ a.e. and $\int f_j^2 dx \rightarrow \int f^2 dx$. Prove $\int |f_j - f|^2 dx \rightarrow 0$.

2. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative, C^∞ function with compact support such that

$$\int_{\mathbb{R}} \varphi(x) dx = 1.$$

Define

$$\varphi_\sigma(x) = \sigma^{-1} \varphi\left(\frac{x}{\sigma}\right) \quad \text{and} \quad u_\sigma(x) = \int \varphi_\sigma(x-y) u(y) dy.$$

For $u \in L^2(\mathbb{R})$, prove

$$\int_{\mathbb{R}} |u_\sigma(x)|^2 dx \leq \int_{\mathbb{R}} |u(x)|^2 dx.$$

3. Assume $f : [0, 1] \rightarrow \mathbb{R}$ is uniformly continuous, increasing and convex. Prove f is differentiable almost everywhere and

$$f(1) - f(0) = \int_0^1 f'(x) dx.$$

4. Assume $f : [0, 1] \rightarrow \mathbb{R}$ is a measurable function such that $fg \in L^1([0, 1])$ for all $g \in L^2([0, 1])$. Prove $f \in L^2([0, 1])$.

5. Let $U \subset \mathbb{C}$ be an open set. Assume $f, g : U \rightarrow \mathbb{C}$ are holomorphic functions such that $\bar{f}g$ is holomorphic. Prove either f is constant or g is identically equal to 0.

6. Assume $f : \mathbb{C} \rightarrow \mathbb{C}$ is a non-constant entire function. Prove $f(\mathbb{C})$ is dense in \mathbb{C} .

7. Prove that $z^5 + 3z^3 + 7$ has all its zeros in the disk $D(0, 2) = \{z \in \mathbb{C} : |z| < 2\}$.

8. Let $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$. Assume $r > 1$ and $f : \overline{D(0, r)} \setminus D(0, 1) \rightarrow \mathbb{C}$ is a continuous function, holomorphic on $D(0, r) \setminus \overline{D(0, 1)}$ that satisfies

$$\max_{\partial D(0, 1)} |f(z)| = 1 \quad \text{and} \quad \max_{\partial D(0, r)} |f(z)| = R.$$

Prove $\log |f(z)| \leq \frac{\log R}{\log r} \log |z|$.

Qualifying Exam - Analysis - Fall 2014

Justify your answers to all problems.

1. Let Q be the unit square in \mathbb{R}^2 . Consider functions $f_n \in L^1(Q)$ such that

$$f_n \rightarrow f \text{ almost everywhere in } Q \text{ and } \int_Q |f_n| \rightarrow \int_Q |f| < \infty.$$

(a) Prove that $\int_A |f_n| \rightarrow \int_A |f|$ for every measurable subset A of Q .

(b) Prove that $f_n \rightarrow f$ in L^1 .

2. Let $f \in L^1(\mathbb{R}^d)$ and M_f denote the Hardy-Littlewood maximal function of f ; in other words,

$$M_f(x) = \sup_B \frac{1}{m(B)} \int_B |f(y)| dy, \quad x \in \mathbb{R}^d$$

where the supremum is taken over all balls containing the point x . Prove that

$$m(\{x : M_f(x) > \alpha\}) \leq \frac{A}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}, \quad \forall \alpha > 0$$

where A is a constant depending only on d and $\|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)| dx$.

3. Let X and Y be Hilbert spaces and $L : X \rightarrow Y$ be a bounded linear operator. Prove that the following two conditions are equivalent:

(a) The image $L(\mathbf{B})$ of the unit ball in X has compact closure in Y .

(b) There is a sequence of bounded linear operators $\{L_n : X \rightarrow Y\}$ such that the image of $L_n(X)$ is finite dimensional and such that $\|L_n - L\| \rightarrow 0$. (Here, $\|\cdot\|$ is the operator norm.)

4. Let $\Omega \subset \mathbb{C}$ be a bounded region and $\{f_n\}$ be a sequence of continuous functions on $\bar{\Omega}$ which are holomorphic in Ω . If $\{f_n\}$ converges uniformly on the boundary of Ω , then prove that f_n converges uniformly on Ω .

5. Compute

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{\pi(a+1)e^{-a}}{4}, \quad a > 0.$$

6. Assume that f and g are entire functions and that g never vanishes. If $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$, then prove that there is a constant C such that $f(z) = Cg(z)$.

7. Let $D \subset \mathbb{C}$ be the unit disk. Prove that every one-to-one conformal mapping of D to D is given by a linear fractional transformation.

QUALIFYING EXAM - ANALYSIS - SPRING 2014

1. Prove the following statement without using Ergoroff's Theorem: Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E , $f_k \rightarrow f$ a.e. on E and there exists $g \in L^1(E)$ such that $|f_k| \leq g$ for all k . Given $\epsilon > 0$, there exists a closed set A_ϵ such that $m(E \setminus A_\epsilon) < \epsilon$ and $f_k \rightarrow f$ uniformly on A_ϵ .

2. Let $f \in L^1(\mathbb{R})$ and define $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}} |f(x)| dx = \int_0^{\infty} m(E_\alpha) d\alpha.$$

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. Prove the following statement: There exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$ if and only if f is absolutely continuous and $|f'| \leq M$.

4. (a) Prove that the operator $T : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined by setting $T[f](x) = xf(x)$ is continuous and symmetric (self-adjoint).

(b) Prove that T is not compact.

5. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D \rightarrow D$ be a holomorphic function. Prove

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}, \quad \forall z \in D.$$

6. For $t \in \mathbb{R}$, compute

$$\lim_{A \rightarrow \infty} \int_{-A}^A \frac{\sin x}{x} e^{ixt} dx.$$

7. Let $U \subset \mathbb{C}$ be an open set, $f : U \rightarrow \mathbb{C}$ be a holomorphic function and $z_0 \in U$. Prove that if $f'(z_0) = 0$, then f is not one-to-one in any neighborhood of z_0 .

8. Prove that if f is an entire function and $|f(z)| \leq a + b|z|^k$ for all $z \in \mathbb{C}$ where a, b and k are positive real numbers, then f is a polynomial.

QUALIFYING EXAM - ANALYSIS - FALL 2013

Justify your answers to all problems.

1. Let I denote the interval $(0, 1)$. Suppose that $f : I \rightarrow \mathbb{R}$ with $\int_0^1 |f(t)| dt < +\infty$. Define $g : I \rightarrow \mathbb{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t} dt.$$

Show that $g \in L^1(I)$.

2. Does there exist a nonempty measurable set $E \subset \mathbb{R}$ satisfying the following two properties:

- (a) given $x, y \in E$, there exists $z \notin E$ that lies between x and y ;
- (b) E has no isolated points?

3. Prove that smooth compactly supported functions are dense in $L^2(\mathbb{R}^n)$.

4. Determine whether there is a nonzero smooth compactly supported function on \mathbb{R} whose Fourier transform is also compactly supported?

5. This problem is about the integral

$$I = \int_0^\infty \frac{\cos u}{u^4 + 1} du.$$

(a) Show directly that I is a convergent improper Riemann integral.

(b) Is

$$\int_{[0, \infty)} \frac{\cos u}{u^4 + 1} d\mu(u)$$

a well-defined Lebesgue integral, where μ denotes the Lebesgue measure on \mathbb{R} ?

(c) (main part) Evaluate the integral in (a).

6. Determine the number of *distinct* solutions of the equation

$$e^{z^2} = 5z^5$$

in the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

7. Determine all entire functions f (i.e., $f(z)$ is holomorphic on the whole z -plane) that satisfy the inequality

$$|f(z)| \leq |z|^2 |\operatorname{Im} z|^2$$

for $|z|$ sufficiently large.

Qualifying Exam - Analysis

May, 2013

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Problem 1 Let $U \subset \mathbb{C}$ be an open set and let f be a continuous function on U . If f^2 is holomorphic on U , prove that f is holomorphic on U .

Problem 2 Prove that there is only one solution in the unit disc $\{z : |z| < 1\}$ and there are three solutions on the annulus $\{z : 1 < |z| < 2\}$ (counting multiplicities) for the equation $z^4 - 6z + 3 = 0$.

Problem 3 Let f be a holomorphic function on the unit disc $\{z : |z| < 1\}$ satisfying $f(0) = 0$ and $\operatorname{Re} f(z) \leq A$ for some positive number $A > 0$. Prove:

$$|f(z)| \leq \frac{2A|z|}{1 - |z|}.$$

Problem 4 Calculate the following integral:

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{4 + x^2} dx.$$

Problem 5 Suppose that E and F are Lebesgue measurable sets of \mathbb{R} , and their Lebesgue measures $m(E) > 0$, $m(F) > 0$. Prove that

$$E + F = \{x + y : x \in E, y \in F\}$$

contains a nonempty open interval.

Problem 6(a) Prove the Riemann-Lebesgue Lemma: if $f \in L^1(\mathbb{R}^d)$, then the Fourier transform of f ,

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \rightarrow 0, \text{ as } |\xi| \rightarrow \infty.$$

(b) Use part (a) to justify whether there exists a function $h \in L^1(\mathbb{R}^d)$ such that

$$f * h = f \text{ for all } f \in L^1(\mathbb{R}^d).$$

Here $f * h$ is the convolution of f and h defined by

$$(f * h)(x) = \int_{\mathbb{R}^d} f(x - y) h(y) dx.$$

Problem 7 If the sequence of Lebesgue measurable functions $\{f_n\}_{n=1}^\infty$ on \mathbb{R}^d satisfying that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} |f_n(x)|^2 dx = 0,$$

show that there exists a subsequence of functions $\{f_{n_j}\}_{j=1}^\infty$ such that

$$f_{n_j}(x) \rightarrow 0 \text{ a.e. } x.$$

Problem 8 Recall that the inner product on $L^2(\mathbb{R}^d)$ is given by

$$(f, g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx, \text{ for } f, g \in L^2(\mathbb{R}^d),$$

which induces the L^2 -norm

$$\|f\|_{L^2} = (f, f)^{1/2}.$$

(a) If the sequence of functions $\{f_n\}_{n=1}^\infty$ in $L^2(\mathbb{R}^d)$ satisfy that $\|f_n\|_{L^2} = 1$, show that there exists a subsequence of functions $\{f_{n_j}\}_{j=1}^\infty$ such that f_{n_j} converges weakly to some function f in $L^2(\mathbb{R}^d)$, i.e.,

$$(f_{n_j}, g) \rightarrow (f, g) \text{ for all } g \in L^2(\mathbb{R}^d).$$

(b) If $f_n \rightarrow f$ weakly in $L^2(\mathbb{R}^d)$ and $\|f_n\|_{L^2} \rightarrow \|f\|_{L^2}$ as $n \rightarrow \infty$, show that $\|f_n - f\|_{L^2} \rightarrow 0$ as $n \rightarrow \infty$.

ANALYSIS QUALIFYING EXAM, FALL 2012

Part I. Complex Analysis.

1. How many zeros does the polynomial

$$z^9 + z^6 + 30z^5 - 3z + 2$$

have in the annulus $\{1 \leq |z| \leq 3\}$. Justify your answer.

2. Let $f(x) = \frac{1}{x^2+1}$. Use residues to compute the Fourier transform

$$\widehat{f}(t) = \int_{-\infty}^{+\infty} f(x)e^{-itx} dx .$$

3. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk.

What is the maximum possible value of $|f'(\frac{1}{2})|$ for a holomorphic function $f : D \rightarrow D$ with $f(\frac{1}{2}) = \frac{3}{4}$? Find all such functions f that attain this maximum value.

4. Let $I = \{t \in \mathbb{R} : 0 \leq t \leq 1\} \subset \mathbb{C}$. Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a continuous function such that f is holomorphic on $\mathbb{C} \setminus I$. Prove that f is an entire function (i.e., f is holomorphic on all of \mathbb{C}).

Part II. Real Analysis.

5. For each natural number n , let $f_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of absolutely integrable functions, and let $f : [0, 1] \rightarrow \mathbb{R}$ be another absolutely integrable function such that

$$\int_0^1 |f_n(x) - f(x)| dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

- (a) Show that there exists a subsequence f_{n_j} of f_n which converges to f pointwise almost everywhere.
- (b) Give a counterexample to show that the assertion fails if "pointwise almost everywhere" is replaced by "uniformly".
6. For this problem, consider just Lebesgue measurable functions $f : [0, 1] \rightarrow \mathbb{R}$. together with the Lebesgue measure.
- (a) State Fatou's lemma (no proof required).
- (b) State and prove the Dominated Convergence Theorem.
- (c) Give an example where $f_n(x) \rightarrow 0$ a.e., but $\int_{-\infty}^{+\infty} f_n(x) dx \rightarrow 1$.

7. Let

$$f * g(x) := \int_{-\infty}^{+\infty} f(y)g(x - y)dy$$

denote the convolution of f and g .

- (a) Let $f, g \in L^2(\mathbb{R})$ be two square-integrable functions on \mathbb{R} (with the usual Lebesgue measure). Show that the convolution $f * g$ bounded continuous function on \mathbb{R} .
- (b) Instead let $h \in L^1(\mathbb{R})$ be fixed. Show that $A(f) = f * h$ is a bounded operator $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$.
8. Let T be a linear transformation on $C_0(\mathbb{R})$, the space of continuous functions of compact support, that has the following two properties:

$$\|Tf\|_{L^\infty} \leq \|f\|_{L^\infty}, \quad \text{and} \quad |\{x \in \mathbb{R} : |Tf(x)| > \lambda\}| \leq \frac{\|f\|_{L^1}}{\lambda}.$$

(Here $|A|$ denotes the Lebesgue measure of the set A .) Prove that

$$\int_{-\infty}^{+\infty} |Tf(x)|^2 dx \leq C \int_{-\infty}^{+\infty} |f(x)|^2 dx$$

for all $f \in C_0(\mathbb{R})$ and some fixed number C .

Analysis Qualifying Exam, May 9, 2012

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours

Part I. Complex Analysis

1. Use residues to calculate the integral $\int_0^\infty \frac{1}{(1+x^2)^2} dx$.
2. Suppose f is holomorphic on the open unit disc $D(0, 1)$ and continuous on $\overline{D(0, 1)}$. Assume $|f(\xi)| < 1$ for $\xi \in \partial D(0, 1)$. Show that there exists a unique point $a \in D(0, 1)$ such that $f(a) = a$.
3. Suppose f is holomorphic on $U := D(0, 1) \setminus \{0\}$. Assume that the real part $\operatorname{Re}(f)$ is bounded from below on U . Prove that $z = 0$ is a removable singularity.
4. Let $U = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \leq \frac{\pi}{2}\}$ and f be an entire function satisfying $f(U) \subset U$, $f(-1) = 0$, $f(0) = 1$. Prove that $f(z) = z + 1$.

Part II. Real Analysis

5. Justify or give a counterexample to the following assertions:
 - a. If $\{f_i\}$ is a sequence in $L^2([0, 1])$ converging weakly to f in $L^2([0, 1])$, then f_i^2 converges weakly to f^2 in $L^1([0, 1])$.
 - b. If $\{f_i\}$ is a sequence in $L^2([0, 1])$ converging strongly to f in $L^2([0, 1])$, then f_i^2 converges strongly to f^2 in $L^1([0, 1])$.
6. Let $\{g_k\}_{k=1}^\infty$ be a sequence in $L^1(\mathbb{R}^n)$ with $\sum \|g_k\|_{L^1(\mathbb{R}^n)} < \infty$.
 - a. Show that $\sum_{k=1}^\infty g_k$ converges a.e. to a function $g \in L^1(\mathbb{R}^n)$.
 - b. Show that $\lim_{N \rightarrow \infty} \|g - \sum_{k=1}^N g_k\|_{L^1(\mathbb{R}^n)} = 0$.
7. Let $f \in L^1(\mathbb{R})$ and set $h(x) = \int_{[x, x+1]} f(t) dt$.
 - a. Show that $h(x)$ is absolutely continuous.
 - b. Show that $\lim_{x \rightarrow \infty} h(x) = 0$.
8. Let $f \in L^1(\mathbb{R})$. Define its Fourier transform $\hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} dx$. Show that $\hat{f}(\xi) \in C_0(\mathbb{R})$, that is the Fourier transform is continuous and vanishes at infinity. *You may not quote the Riemann-Lebesgue lemma without sketching a proof.*

**ANALYSIS QUALIFYING EXAM
SEPTEMBER 2011**

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$

1. Determine the value of the integral

$$\int_{\gamma} \frac{dz}{z^3 \cos z},$$

where γ is the circle $\{|z - 1| < 2\}$ traversed counterclockwise.

2. Let $h : \mathbb{C} \rightarrow \mathbb{R}$ be a harmonic function such that h is bounded below. Prove that h is constant.
3. Let f be a holomorphic function on $D \setminus \{0\}$. Suppose that there exists a positive integer n such that $f^{-1}(w)$ contains at most n points for all $w \in \mathbb{C}$. Prove that 0 is a removable singularity or pole.
4. Suppose that U is a simply connected bounded domain in \mathbb{C} , and let $P \in U$. Prove that for all $t \in \mathbb{R}$, there exists a unique holomorphic function $f : U \rightarrow U$ such that $f(P) = P$ and $f'(P) = e^{it}$.

Part II. Real Analysis.

Notation: $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$.

5. Give an example of a sequence of functions $\{f_j\}$ satisfying $\|f_j\|_{L^2(\mathbb{R})} = 1$ for which $\{f_j\}$ has no convergent subsequence in $L^2(\mathbb{R})$.
6. a) Let $f_j \in L^2(\mathbb{R}^n)$ and suppose that

$$\int_{\mathbb{R}^n} |f_j(x) - f(x)|^2 dx \rightarrow 0.$$

If $\Omega \subset \mathbb{R}^n$ has finite Lebesgue measure, i.e., $|\Omega| < \infty$, show that the Fourier transforms satisfy

$$\int_{\Omega} \widehat{f_j}(\xi) d\xi \rightarrow \int_{\Omega} \widehat{f}(\xi) d\xi. \quad (1)$$

- b) If $|\Omega| = \infty$, is (1) still always valid? Give a proof or counterexample.
7. Let $\omega(\alpha) = |\{x : |f(x)| > \alpha\}|$, $\alpha > 0$, be the distribution function of a given $f \in L^p(\mathbb{R}^n)$, where $p > 0$. Does $\alpha^p \omega(\alpha)$ tend to a limit as $\alpha \rightarrow 0+$? Give a proof or counterexample.
8. Show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that

$$f * I = f \quad \text{for all } f \in L^1(\mathbb{R}^n).$$

(Here $(f * I)(x) = \int f(y) I(x - y) dy$ is the convolution of f and I .)

ANALYSIS QUALIFYING EXAM
MAY 2011

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$

1. Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$. Give explicit formulas for the functions and give a proof for your answer. (An entire function is a holomorphic function on \mathbb{C} .)
2. Let $f : D \rightarrow \mathbb{C}$ be a holomorphic function with simple zeros at the points $1/3$, $2/3$, $i/4$ and no other zeros. Determine the value of the integral

$$\int_{\{|z|=1/2\}} (z^2 - 1) e^z \frac{f'(z)}{f(z)} dz,$$

where the direction of integration is counterclockwise.

3. Let U be a bounded domain in \mathbb{C} , and let $f : U \rightarrow U$ such that f is holomorphic. Let $P \in U$ and suppose that $f(P) = P$. Prove that $|f'(P)| \leq 1$.

Hint: Consider the sequence of iterates $f_n = f \circ f \circ \cdots \circ f$ (n times).

4. Suppose that $u : \mathbb{C} \rightarrow \mathbb{R}$ is a harmonic function such that

$$u(z) \leq 10 \log(|z| + 2),$$

for all $z \in \mathbb{C}$. Prove that u is constant.

Part II. Real Analysis.

5. Let $f_n : [0, 1] \rightarrow \mathbb{R}$, for $n = 1, 2, \dots$, be a sequence of \mathcal{C}^1 functions such that $f_n(t) \leq 5$ and $|f'_n(t)| \leq 1$ for all n, t . Define the functions $g_n : [0, 1] \rightarrow \mathbb{R}$ by

$$g_n(t) = \max\{f_1(t), \dots, f_n(t)\}$$

for $n = 1, 2, \dots$. Prove that the sequence $\{g_n\}$ converges uniformly on $[0, 1]$.

6. Let $f \in L^1(S^1)$ such that $\widehat{f} \in \ell^1(\mathbb{Z})$. Prove that $f \in \mathcal{C}(S^1)$ (continuous functions on the circle S^1).

7. Suppose that $f \in L^\infty([0, 1])$.

a) Prove that if $1 < p < \infty$ then $\|f\|_p \leq \|f\|_\infty$.

b) Show that $\|f\|_\infty \leq \lim_{p \rightarrow \infty} \|f\|_p$ and therefore conclude that $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.

Hint: Given $\varepsilon > 0$, consider $A_\varepsilon = \{x \in [0, 1] : |f(x)| > \|f\|_\infty - \varepsilon\}$.

8. a) Let $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$, for $j = 1, 2, \dots$, be a sequence of L^2 functions. Suppose that there is a function $f \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f_j g \rightarrow \int_{\mathbb{R}^n} f g, \quad \forall g \in L^2(\mathbb{R}^n).$$

Show that

$$\|f\|_2 \leq \liminf_{j \rightarrow \infty} \|f_j\|_2.$$

Also, give an example showing that strict inequality can occur.

- b) Suppose also that $\|f_j\|_2 \rightarrow \|f\|_2$. Show that in this case $\|f_j - f\|_2 \rightarrow 0$ as $j \rightarrow \infty$.

Instructions: Do all eight problems. Each problem will be scored out of 10 points.

1. Suppose that $f_j \in L^2(\mathbb{R}^n)$, $j = 1, 2, 3, \dots$ and that $f_j \rightarrow f$ in L^2 . Suppose further that there is a constant $M < \infty$ so that

$$\int e^{100|x|^2} |f_j(x)|^2 dx \leq M, \quad j = 1, 2, 3, \dots$$

Is it true that $\int e^{99|x|^2} |f(x)|^2 dx < \infty$? Give a proof or counterexample.

2. Let $E, F \subset \mathbb{R}$ be two Lebesgue-measurable subsets of \mathbb{R} , each of finite measure, and let χ_E and χ_F denote their respective characteristic functions.

(a) Prove that the convolution $\chi_E * \chi_F$ defined by

$$\chi_E * \chi_F(x) = \int_{\mathbb{R}} \chi_E(y) \chi_F(x - y) dy$$

is a continuous function of x .

(b) Show that as $n \rightarrow \infty$,

$$n(\chi_E * \chi_{[0,1/n]}) \rightarrow \chi_E$$

pointwise almost everywhere.

3. Let $Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$, where $K(x, y)$ is a nonnegative measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that there are measurable functions $p(x) > 0$ and $q(x) > 0$ on \mathbb{R}^n and real numbers $\alpha, \beta > 0$ for which

$$\int K(x, y) q(y) dy \leq \alpha p(x),$$

for almost all x and

$$\int p(x) K(x, y) dx \leq \beta q(y)$$

for almost all y . Show that for $f \in L^2(\mathbb{R}^n)$ we have

$$\|Tf\|_{L^2} \leq \sqrt{\alpha\beta} \|f\|_{L^2}.$$

(This is called *Schur's test*.)

4. Define $U : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$Uf(x) = f(x - 1).$$

Show that if $f \in L^2$ satisfies $Uf = \lambda f$, for some $\lambda \in \mathbb{R}$ (i.e., f is an eigenvector of U) then f must be the zero element, i.e., $f = 0$ almost everywhere.

[cont'd on other side]

5. Let γ be the closed curve in the complex plane that is given in polar coordinates by $r = 2 + 3 \cos \theta$, $0 \leq \theta \leq 4\pi$, oriented in the direction of increasing θ . Let

$$f(z) = \frac{e^z}{2z - 1} + \frac{\sin(2z)}{(z - 2)^2} + \frac{\cos(5z)}{(z + 5i)^3}.$$

Calculate $\int_{\gamma} f(z) dz$.

[Recall that in polar coordinates, $(-r, \theta)$ and $(r, \theta + \pi)$ give the same point in the plane.]

6. Let D denote the open unit disc in \mathbb{C} . Let $f : D \rightarrow \mathbb{C}$ be a C^1 function, and consider the property: f has a double zero at $z = \frac{1}{n}$ for all natural numbers n .

(a) Determine all holomorphic functions f with this property. [The terms “holomorphic” and “complex analytic” have the same meaning.]

(b) Give an example of a *non*-holomorphic C^1 function with this property. (You must explain why your example has this property.)

7. Determine all entire functions f (i.e., $f(z)$ is holomorphic and is defined for all $z \in \mathbb{C}$) that satisfy the inequality:

$$|f(z)| \leq |\operatorname{Re} z|^2 + |z|^{\frac{3}{2}} \quad \text{whenever } |z| > 1.$$

8. Let D denote the open unit disc, as in #6. Let $g : D \rightarrow D$ be a surjective holomorphic mapping for which $g(0) = 0$. Suppose that $z = g(w)$ gives a two-sheeted branched covering of the image with exactly one branch point at $w = 0$. An example of such a function g is $g(w) = w^2$.

(a) Express the given conditions explicitly in terms of g and its derivatives.

(b) Show that $|g(w)| \leq |w|^2$ for all $|w| < 1$.

(c) Suppose that $g(1/2) = i/4$. What is the strongest statement about $g(w)$ that follows from the assertion in (b)? **Explain.**

**ANALYSIS QUALIFYING EXAM
MAY 2010**

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Complex Analysis.

1. Let f be a holomorphic function on the punctured disk

$$U := \{z \in \mathbb{C} : 0 < |z| < 1\} .$$

Suppose that $|f(z)| \leq |z|^{-1/2}$ for all $z \in U$. Prove that f has a removable singularity at 0.

2. Find all possible values of

$$\int_{\gamma} \frac{e^{\pi z}}{(z-1)(z-i)^2} dz$$

where γ ranges over all simple closed smooth curves contained in $\mathbb{C} \setminus \{1, i\}$. (A simple closed curve is a closed curve that does not intersect itself; i.e., it is a homeomorphic image of the circle.)

You do not need to give a proof for your answer to this problem, but show all your work.

3. Let $\mathcal{O}(D)$ denote the space of holomorphic functions on the unit disk D and let

$$\mathcal{H} = \mathcal{O}(D) \cap L^2(D) = \left\{ f \in \mathcal{O}(D) : \int_D |f|^2 dx dy < +\infty \right\} .$$

- a) Show that for all compact sets $K \subset D$, there is a constant $C_K \in \mathbb{R}^+$ such that

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{L^2(D)} .$$

- b) Show that \mathcal{H} is a closed subspace of $L^2(D)$ and hence is a Hilbert space.

4. Let h be a harmonic function on the domain

$$U := \{z \in \mathbb{C} : |z| > 1\} .$$

Show that there exists a constant $c \in \mathbb{R}$ and a holomorphic function f on U such that $\operatorname{Re} f(z) = h(z) + c \log |z|$ for all $z \in U$.

Part II. Real Analysis.

5. Let $f_j \in L^2(\mathbb{R}^n)$, and \widehat{f}_j denote its Fourier transform for $j = 1, 2, 3, \dots$. Suppose that $f_j \rightarrow f$ in L^2 and that there is a finite constant M so that

$$\|f_j\|_{H^\sigma} \leq M, \quad j = 1, 2, 3, \dots,$$

for some $\sigma \in \mathbb{R}$, where $\|g\|_{H^\sigma} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^\sigma |\widehat{g}(\xi)|^2 d\xi\right)^{1/2}$ denotes the H^σ Sobolev norm of g . Is it necessarily true that $\|f\|_{H^\sigma} < \infty$? Give a proof or counterexample.

6. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function with compact support.
- a) Prove that if $1 \leq p \leq q \leq \infty$ are fixed then there is a constant A such that

$$\|f * \varphi\|_{L^q} \leq A\|f\|_{L^p}, \quad \text{for all } f \in L^p.$$

If you use Young's (convolution) inequality, you should prove it.

- b) Show by example that such a general inequality cannot hold for $p > q$.

7. Suppose that

$$f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$$

is continuous and has the property that for each x the map $t \rightarrow f(x, t)$ is differentiable and that $|\frac{\partial f}{\partial t}(x, t)| \leq g(x)$ for some measurable function satisfying $\int_0^1 g(x) dx < \infty$. Carefully prove that $F(t) = \int_0^1 f(x, t) dx$ satisfies

$$F'(t) = \int_0^1 \frac{\partial f}{\partial t}(x, t) dx.$$

8. Let E be a measurable subset of the line.

- a) Let $\chi_E : \mathbb{R} \rightarrow \mathbb{R}$ be the characteristic function of E (i.e. $\chi_E(x) = 1$ when $x \in E$ and $\chi_E(x) = 0$ when $x \notin E$). If E has finite Lebesgue measure, show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \int_{\mathbb{R}} \chi_E(y) \chi_E(y - x) dy$$

is continuous.

- b) Suppose instead that E has positive Lebesgue measure $0 < |E| \leq \infty$. Using a), show that the set $E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

**ANALYSIS QUALIFYING EXAM
SEPTEMBER 2009**

All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

Part I. Real Analysis. Do 3 out of the following 4 problems.

1. Suppose f_n is a sequence of continuous functions on $[0, 1]$ which converges to a continuous function f on $[0, 1]$. Does it follow that f_n converge uniformly? Give a proof or provide a counterexample.
2. For which values of $\sigma \in \mathbb{R}$ does there exist a constant $C_\sigma < +\infty$ such that

$$\left| \sum_{j,k=1}^{\infty} (1 + |j - k|)^\sigma a_j b_k \right| \leq C_\sigma \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |b_k|^2 \right)^{1/2}$$

Prove your assertion.

3. Let I be the unit interval $[0, 1]$, and for $n = 1, 2, 3, \dots$ and $0 \leq j \leq 2^n - 1$ let

$$I_{n,j} = [j2^{-n}, (j+1)2^{-n}].$$

For $f \in L^1(I, dx)$ define $E_n f(x) = \sum_{j=0}^{2^n-1} (2^n \int_{I_{n,j}} f dt) \chi_{I_{n,j}}(x)$, where $\chi_{I_{n,j}}$ is the characteristic function of $I_{n,j}$. Prove that if $f \in L^1(I, dx)$ then $\lim_{n \rightarrow \infty} E_n f(x) = f(x)$ almost everywhere in I .

4. Let $f(x)$ be a non-decreasing function on $[0, 1]$. You may assume that f is differentiable almost everywhere. Prove that

$$\int_0^1 f'(x) dx \leq f(1) - f(0).$$

Part II. Complex Analysis. Do 3 out of the following 4 problems.

5. Let

$$f(x + iy) = x^3 - 3xy^2 + iy^3.$$

State whether each of the following is true or false and give proofs for your answers:

- a) the complex derivative $f'(0)$ exists;
- b) f is holomorphic in a neighborhood of 0.

6. Let

$$f(z) = \frac{z}{\tan z} \quad \text{for } z \neq 0.$$

- a) Prove that f has a removable singularity at 0.
- b) What is the radius of convergence of the power series for f centered at 0? Justify your answer.

7. Let $f : H \rightarrow D$ be a holomorphic map from the upper half plane

$$H = \{z \in \mathbb{C} : \text{Im } z > 0\} \text{ to the unit disk } D = \{z \in \mathbb{C} : |z| < 1\}.$$

Suppose that $f(i) = 1/2$. Determine the maximal possible value of $|f'(i)|$.

8. Let h be a harmonic function on the punctured disk

$$U := \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Show that there exists a constant $c \in \mathbb{R}$ and a holomorphic function f on U such that $\text{Re}f(z) = h(z) + c \log |z|$ for all $z \in U$.

**ANALYSIS QUALIFYING EXAM
MAY 2009**

Do all 8 problems. All problems are equally weighted. Show all your work. In each solution, state any theorems you are applying and verify that the hypotheses are satisfied.

Time: 3 hours.

1. Find all meromorphic functions f on \mathbb{C} such that

$$|f(z)| \leq \frac{\log(2 + |z|^2)}{|z|} \quad \text{for all } z \neq 0.$$

Give explicit formulas for the functions and give a proof for your answer.

2. How many solutions does the equation

$$z + e^{-z} = 2 + i$$

have in the half-plane $\operatorname{Re} z > 0$? Prove that your answer is correct.

3. Let $f_n : U \rightarrow \mathbb{C}$, $n = 1, 2, 3, \dots$, be a sequence of holomorphic functions such that $f_n^{-1}(0) = \{c_n\}$, where $c_n \in U$, and U is a connected open set. Suppose that $f_n \rightarrow f_0$ uniformly, where f_0 is not constant.

a) Prove that f_0 has at most one zero in U .

b) Can f_0 have no zeros? If so, give a necessary and sufficient condition on the c_n for this to happen.

4. Let $f(x) = \frac{1}{x^2 + 1}$. Use a contour integral consisting of the interval $[-R, R] \subset \mathbb{R}$ and a semicircle of radius R to compute the Fourier transform

$$\widehat{f}(1) = \int_{\mathbb{R}} f(x)e^{-ix} dx.$$

Show that the contour integral converges to your answer as $R \rightarrow +\infty$.

5. Let $f, g \in L^2(\mathbb{R})$ be two square-integrable functions on \mathbb{R} (with the usual Lebesgue measure). Show that the convolution

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x - y) dy$$

of f and g is a bounded continuous function on \mathbb{R} .

6. Let \mathbb{R}/\mathbb{Z} be the unit circle with the usual Lebesgue measure. For each $n = 1, 2, 3, \dots$ let $K_n : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}_+$ be a nonnegative integrable function such that $\int_{\mathbb{R}/\mathbb{Z}} K_n(t) dt = 1$ and $\lim_{n \rightarrow \infty} \int_{\varepsilon \leq |t| \leq 1/2} K_n(t) dt = 0$ for every $0 < \varepsilon < 1/2$, where we identify \mathbb{R}/\mathbb{Z} with $(-1/2, 1/2]$ in the usual way. (Such a sequence of K_n are called *approximations to the identity*.) Let $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ be continuous, and define the convolutions $f * K_n : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ by

$$f * K_n(x) = \int_{\mathbb{R}/\mathbb{Z}} f(x-t)K_n(t) dt.$$

Show that $f * K_n$ converges uniformly to f .

7. Fix $1 \leq p < \infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions $f_n : [0, 1] \rightarrow \mathbb{C}$. Suppose there exists $f \in L^p([0, 1])$ such that $f_n \rightarrow f$ in L^p , that is,

$$\int_{[0,1]} |f_n(x) - f(x)|^p dx \rightarrow 0.$$

- a) Show that $f_n \rightarrow f$ in measure, that is,

$$\lim_{n \rightarrow \infty} \mu(\{x \in [0, 1] : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$$

for all $\varepsilon > 0$. (Here μ = Lebesgue measure.)

- b) Show that there is a subsequence f_{n_k} such that $f_{n_k}(x) \rightarrow f(x)$ almost everywhere.

8. Consider $[0, 1]$ with Lebesgue measure. Let $f \in L^\infty([0, 1])$ and define

$$a_n = \int_{[0,1]} |f|^n dx.$$

Show that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty.$$

PROBLEMS FOR ANALYSIS QUALIFYING EXAM
Fall 2008

Do all seven problems. Show all work and state any theorems you are using.
Time: 3 hours.

1) (15 points) Consider the mapping $F : [0, 1] \rightarrow [0, 1]$ given by $F(s) = s^2$.

Let $F^{-j}(A)$ be the inverse image of j iterates of F applied to a measurable subset $A \subset [0, 1]$. That is, if $F = F^1$ and $F^j, j = 2, 3, \dots$ is defined inductively as $F^j = F^{j-1} \circ F$, then $F^{-j}(A) = \{x : F^j x = y, \text{ some } y \in A\}$.

a) Given $N = 1, 2, \dots$ show that $\mu_N(A) = N^{-1} \sum_{j \leq N} |F^{-j}(A)|$ is a measure which is absolutely continuous with respect to Lebesgue measure. Here $|B|$ denotes the Lebesgue measure of a measurable set.

b) Show that $\mu_N([a, b]) \rightarrow 0$ if $0 < a < b \leq 1$.

c) If f is a continuous function on $[0, 1]$ does $\lim \int_{[0,1]} f(s) d\mu_N(s)$ tend to a limit? If so, what is the limit?

2) (10 points) Let (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) be σ -finite measure spaces and let $K(x, y)$ be a measurable function with respect to the product σ -algebra $\mathcal{M} \times \mathcal{N}$. Assume that there is a constant $0 < A < \infty$ so that for all $x \in X$

$$\int_Y |K(x, y)| d\nu(y) \leq A,$$

and for all $y \in Y$,

$$\int_X |K(x, y)| d\mu(x) \leq A.$$

Let $1 \leq p \leq \infty$ and for $f \in L^p(X, \mathcal{M}, \mu)$ define

$$Tf(y) = \int_X f(x) K(x, y) d\mu(x).$$

Prove that

$$\|TF\|_{L^p(\nu)} \leq A\|f\|_{L^p(\mu)}.$$

3) (10 points) Is the Banach space ℓ^∞ of bounded complex sequences $a = \{a_n\}_{n=1}^\infty$ with the supremum norm $\|a\|_\infty = \sup_n |a_n|$ separable? Prove your assertion.

4) (10 points) Use residues to verify that

$$\int_0^\infty \frac{\ln x}{(x^2 + 4)^2} dx = \frac{\pi}{32} (\ln 2 - 1).$$

5) (10 points) How many solutions does the equation

$$e^z = 3z^7$$

have in the unit disk $D = \{x \in \mathbb{C} : |z| < 1\}$? Justify your answer.

6) (10 points) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. Prove that if there exists some real number C and some positive integer k so that

$$|f(z)| \leq C|z|^k$$

for all z with $|z| > 1$, then f is a polynomial in z of degree at most k .

7) (10 points) Let $D \subset \mathbb{C}$ be the unit disk and $\Omega \subset \mathbb{C}$ a bounded, simply connected domain. If $f_1 : D \rightarrow \Omega$ and $f_2 : D \rightarrow \Omega$ are holomorphic bijections so that $f_1(0) = f_2(0)$, then how are f_1 and f_2 related to each other?

**PROBLEMS FOR ANALYSIS QUALIFYING EXAM
SPRING 2008**

Do all eight problems. Show all work and state any theorems you are using. Time: 3 hours.

1) Let E, F be two Lebesgue measurable subsets of \mathbb{R} of finite measure, and let χ_E, χ_F be their respective characteristic functions.

a) Show that the convolution $\chi_E * \chi_F$ defined by

$$\chi_E * \chi_F(x) = \int_{\mathbb{R}} \chi_E(y)\chi_F(x-y) dy$$

is a continuous function.

b) Show that

$$n(\chi_E * \chi_{[0,1/n]}) \rightarrow \chi_E$$

as $n \rightarrow \infty$ pointwise almost everywhere.

2) Consider $L^\infty([0, 1])$.

a) If f belongs to this space prove that

$$\lim_{p \rightarrow \infty} \left(\int_0^1 |f|^p dx \right)^{1/p} = \|f\|_\infty.$$

b) Give an example showing that this is false if we replace $L^\infty([0, 1])$ by $L^\infty(\mathbb{R})$.

3) Assume that f is a continuously differentiable 2π periodic function on \mathbb{R} . Show that the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

is absolutely convergent for every t (here $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} dt$).

4) Let ℓ^2 be the space of all square-summable sequences of complex numbers, and let $T : \ell^2 \rightarrow \ell^2$ be a linear operator. Let e_n be the sequence

$$e_n = (00 \cdots 010 \cdots),$$

where 1 is in the n -th position. Let $a_{mn} = \langle Te_m, e_n \rangle$ be the “matrix coefficients” of T .

a) Assume that $\sum_{n,m=1}^{\infty} |a_{mn}|^2 < \infty$. Show that T is a bounded operator on ℓ^2 .

b) Assume instead that $\sup\{|a_{mn}| : 1 \leq n, m < \infty\}$ is finite. Must T be bounded? Explain.

5) Prove the following statement: If f and g are entire functions, $g(z) \neq 0$ and $|f(z)| \leq |g(z)|$ for all $z \in \mathbf{C}$, then $f(z) = Cg(z)$ for some constant C .

6) Let $D = \{z \in \mathbf{C} : |z| < 1\}$ and P and Q be distinct points in D . Prove the following statement: If f and g are conformal (or equivalently biholomorphic) self-maps of D , $f(P) = g(P)$ and $f(Q) = g(Q)$, then $f \equiv g$.

7) Let $U \subset \mathbf{C}$ be an open set, $P \in U$ and f a holomorphic function defined on U so that $f(P) = f'(P) = 0$. Use the Argument Principle to prove the following statement: There exists $\delta > 0$ so that if $0 < |Q| < \delta$, then $f^{-1}(Q)$ contains at least two points.

8) Let $U \subset \mathbf{C}$ be an open set and $P \in U$. Let \mathcal{F} be a family of holomorphic functions from U into the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ that take P to 0.

(a) Show that $\sup\{|f'(P)| : f \in \mathcal{F}\} < \infty$.

(b) Show that there exists a sequence $\{f_n\} \subset \mathcal{F}$ and a holomorphic function $f_0 : U \rightarrow D$ so that $\{f_n\}$ converges uniformly to f_0 on every compact subset of U and $f'_0(P) = \sup\{|f'(P)| : f \in \mathcal{F}\}$.

**ANALYSIS QUALIFYING EXAM
FALL 2007**

- (1) Is the function

$$f(x, y) = x^3 + 3xy^2 - 3x^2y - 10 + i(y^3 + 3x^2y - 3y^2x + 5)$$

complex analytic? Prove that your answer is correct.

- (2) Find all entire analytic functions satisfying $|f(z)| \leq |e^z|$ for all $z \in \mathbb{C}$.
- (3) Let A be the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Let f be a non-constant holomorphic function in a neighborhood of A , and suppose that $|f(z)| = 1$ on ∂A (the boundary of A). Prove that f has at least 2 zeros in A .
- (4) Use the residue calculus to compute $\int_0^\infty \frac{dx}{1+x^n}$.

- (5) Give examples of functions f and g on \mathbb{R} so that $f \in L^1 \setminus L^2$ and $g \in L^2 \setminus L^1$.
- (6) Does there exist an open dense subset of \mathbb{R} with Lebesgue measure equal to one? Either construct an example or prove that one does not exist.
- (7) Let f_n be a sequence of measurable real-valued functions on $[0, 1]$ with

$$\sum_{n=1}^{\infty} \left(\int_0^1 |f_n| \right) \leq 1.$$

Prove that f_n converges to zero almost everywhere.

- (8) Suppose that f and g are $L^1(\mathbb{R})$ functions with compact support and let h be the convolution $f \star g$ (i.e., $h(x) = \int f(x-y)g(y) dy$). Prove that h is uniformly continuous.

**PROBLEMS FOR ANALYSIS QUALIFYING EXAM
SPRING 2007**

- (1) How many zeros does the polynomial $z^6 - 2z^5 + 7z^4 + z^3 - z + 1$ have in the open unit disc $D = \{z : |z| < 1\}$?
- (2) Calculate the integral $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, where $0 < a < 1$.
- (3) Let $f : D \rightarrow D$ be a holomorphic map of the unit disc with $f(0) = 0$, and suppose that f is *not* a rotation (a rotation is a map $r_\theta(z) = e^{i\theta}z$). Let $w \in D$ and consider the sequence $\{w_n\}$ defined by $w_{n+1} = f(w_n)$. Show: $\lim_{n \rightarrow \infty} w_n = 0$.
- (4) Does there exist a surjective holomorphic map $f : D \rightarrow \mathbb{C}$ from the unit disc to the whole complex plane? Prove that your answer is correct.
- (5) For which p 's is the function $1/x$ in $L^p(0, \infty)$?
- (6) Suppose that $f_n : [0, 1] \rightarrow \mathbb{R}$ is a sequence of L^4 functions with $\int f_n^4 \leq 1$ for every n and so that $\lim_{n \rightarrow \infty} \int |f_n| = 0$. Show that f_n goes to 0 weakly in L^4 .
- (7) Suppose that f_n is a sequence of functions in $L^2(\mathbb{R})$ that converges weakly in L^2 to a function $f \in L^2(\mathbb{R})$. Is it possible to have

$$\lim_{n \rightarrow \infty} \|f_n\|_{L^2} = \infty?$$

- (8) Suppose that $f \in L^1(\mathbb{R})$ and $\hat{f}(z) = \int_{\mathbb{R}} e^{-ixz} f(x) dx$. Show that f and \hat{f} cannot both have compact support (except if f is identically zero).